

ON p -ADIC ISING-VANNIMENUS MODEL ON AN ARBITRARY ORDER CAYLEY TREE

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ABSTRACT. In this paper, we continue an investigation of the p -adic Ising-Vannimenus model on the Cayley tree of an arbitrary order k ($k \geq 2$). We prove the existence of p -adic quasi Gibbs measures by analyzing fixed points of multi-dimensional p -adic system of equations. We are also able to show the uniqueness of translation-invariant p -adic Gibbs measure. Finally, it is established the existence of the phase transition for the Ising-Vannimenus model depending on the order k of the Cayley tree and the prime p . Note that the methods used in the paper are not valid in the real setting, since all of them are based on p -adic analysis and p -adic probability measures.

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1. INTRODUCTION

At the first place, J. Vannimenus started to study the Ising model with nearest-neighbor and next-nearest-neighbor interactions on the Cayley tree of order two in the paper [57]. The similar results were numerically obtained for an arbitrary order Cayley tree in [16]. The p -adic counterpart of the Ising-Vannimenus model on the Cayley tree of order two was first studied in [47]. There, it was proposed a measure-theoretical approach to investigate the model in the p -adic setting. The proposed methods have been based on p -adic probability measures. In this paper we consider the Ising-Vannimenus model on an arbitrary order Cayley tree in the p -adic setting¹. Note that the Cayley tree or Bethe lattices (see [50]) were fruitfully used, providing a deeper insight into the behavior of the models. Moreover, they will provide more information about the models defined on complex networks [13].

In the present paper we use the methods based on p -adic probability measures. We point out that the p -adic numbers appeared in quantum physics models (see [7]) such a way the model is described by p -adic probability measures. There are also many models in physics cannot be described using ordinary Kolmogorov's probability theory (see [28, 36, 38, 60]). These papers stimulated the development of the p -adic probability models [8],[27, 29, 34, 52]. Moreover, an abstract non-Archimedean measure theory developed in [31, 37] which laid on the base of the theory of stochastic processes with values in p -adic and more general non-Archimedean fields. Hence, this theory of stochastic processes allowed us to construct wide classes of processes using finite dimensional probability distributions. Therefore, in [18, 42, 44, 48, 54] it has been developed a p -adic statistical mechanics models with nearest neighbor interactions, based on the theory of p -adic probability measures and processes. Namely, we have studied p -adic Ising

¹Note that p -adic numbers provide a more exact and more adequate description of microworld phenomena. Therefore, there are many papers are devoted to the description of various models in the language of p -adic analysis (see for example [1, 3, 4],[15],[27, 28, 32, 33],[38],[58, 59, 60]).

and Potts models with nearest neighbor interactions on Cayley trees. Note that there are also several p -adic models of complex hierarchic systems [32, 33]. We remark that one of the central problems of such a theory is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian, and a description of the set of such measures. In most cases such an analysis depend on a specific properties of Hamiltonian, and complete description is often a difficult problem. This problem, in particular, relates to a phase transition of the model (see [19]).

In this paper, we consider a p -adic analogue of the model [57] on the Cayley tree, i.e. such a model has nearest neighbor and next nearest neighbor interactions. Here, we continue an investigation of the p -adic Ising-Vanniminus model on the Cayley tree of an arbitrary order. We are going to use a measure-theoretical approach proposed in [47]. In the previous study, the order of the Cayley tree being two was essentially useful. In this study, we are going to apply different techniques which will allow us to prove the uniqueness of translation-invariant p -adic Gibbs measure. This partially confirms the conjecture formulated in [47] to be true. Moreover, we will also show the existence of the phase transition depending on the order k of the Cayley tree and prime p . We point out that in the p -adic setting there are several kinds of phase transitions such as *strong phase transition*, *phase transition* (see [42, 43]). Here, by the *phase transition* we mean the existence of at least two non-trivial p -adic quasi Gibbs measures such that one is bounded and the second one is unbounded (note that in the p -adic probability, unlike to real setting, the probability measures could be even unbounded [52]). The reader should refer to [14] for the recent development of the subject.

The present paper is organized as follows. In section 2, we recall some necessary results from the p -adic analysis. In Section 3, we recall definitions of the Ising-Vanniminus model and corresponding p -adic quasi Gibbs measures via interacting functions. Note that such kind of measures exist if the interacting functions satisfy multi-dimensional recurrence equations. In section 4, the existence of p -adic quasi Gibbs measures is established in terms of the order of the tree k and the prime p . To prove such an existence result we are going to investigate fixed points of a multi-dimensional functional equation. In Section 5, we will show that the translation-invariant p -adic Gibbs measure is unique. In particular, the obtained result confirms the conjecture formulated in [47] to be true. In the final section, we are able to establish the existence of the phase transition for the model depending on the order k of the Cayley tree and the prime p . Note that the values of the norms (i.e. "absolute values"), in the p -adic setting, are discrete, therefore it is impossible to determine exact values of a critical point for the existence of the phase transition. Moreover, in the field of p -adic numbers there is no reasonable order compatible with the usual order in the rational numbers, and this rises some difficulties in the direction of determination of exact critical points. We stress that the methods used in the paper are not valid in the real setting, since all of them are based on p -adic analysis and p -adic probability measures.

2. PRELIMINARIES

2.1. p -adic numbers. In what follows p will be a fixed prime number. The set \mathbb{Q}_p is defined as a completion of the rational numbers \mathbb{Q} with respect to the norm $|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{R}$ given by

$$(2.1) \quad |x|_p = \begin{cases} p^{-r} & x \neq 0, \\ 0, & x = 0, \end{cases}$$

here, $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z}$, $n \in \mathbb{N}$, $(m, p) = (n, p) = 1$. The absolute value $|\cdot|_p$ is non-Archimedean, meaning that it satisfies the strong triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. We recall a nice property of the norm, i.e. if $|x|_p > |y|_p$ then $|x + y|_p = |x|_p$. Note that this is a crucial property which is proper to the non-Archimedeanity of the norm.

Any p -adic number $x \in \mathbb{Q}_p$, $x \neq 0$ can be uniquely represented in the form

$$(2.2) \quad x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \dots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p - 1$, $x_0 > 0$, $j = 0, 1, 2, \dots$. In this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called *the k^{th} residue modulo p* if the congruent equation $x^k \equiv a \pmod{p}$ has a solution $x \in \mathbb{Z}$.

By $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ we denote a subgroup of \mathbb{Q}_p . Recall that $\sqrt[k]{-1}$ exists in \mathbb{F}_p whenever -1 is the k^{th} residue of module p . Otherwise, it is said that $\sqrt[k]{-1}$ does not exist in \mathbb{F}_p . Let $\mathcal{N}_{k,p}(\mathbb{F}_p)$ be the number of solutions $x^k \equiv -1 \pmod{p}$ in \mathbb{F}_p . It is known [55] that $\sqrt[k]{-1}$ exists in \mathbb{F}_p if and only if $\frac{p-1}{(k,p-1)}$ is even. Moreover, in this case, one has that $\mathcal{N}_{k,p}(\mathbb{F}_p) = (k, p-1)$. Similarly, we say that $\sqrt[k]{-1}$ exists in \mathbb{Q}_p whenever the equation $x^k = -1$ is solvable in \mathbb{Q}_p . Otherwise, it is said that $\sqrt[k]{-1}$ does not exist in \mathbb{Q}_p . It was shown [49] that $\sqrt[k]{-1}$ exists in \mathbb{Q}_p if and only if $\sqrt[q]{-1}$ exists in \mathbb{F}_p , where $k = q \cdot p^s$ and $(q, p) = 1$ with $s \geq 0$.

For each $a \in \mathbb{Q}_p$, $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$$

and the set of all p -adic integers

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

The set $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ is called a set of p -adic units.

The following lemma is known as the Hensel's lemma

Lemma 2.1. [5, 20] *Let $\mathbf{x} = (x_1, \dots, x_m)$, $\Theta = (0, \dots, 0)$ and $F(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$ be a polynomial function whose coefficients are p -adic integers. Let $\mathbb{J}(F(\mathbf{x}))$ be the Jacobian matrix of the function $F(\mathbf{x})$. If there exists a vector $\mathbf{a} = (a_1, \dots, a_m)$ with p -adic integer components such that*

$$F(\mathbf{a}) \equiv \Theta \pmod{p} \quad \text{and} \quad \det(\mathbb{J}(F(\mathbf{a}))) \not\equiv 0 \pmod{p}$$

then $F(\mathbf{x})$ has a unique root \mathbf{x}_0 with p -adic integer components which satisfies $\mathbf{x}_0 \equiv \mathbf{a} \pmod{p}$.

Recall that the p -adic logarithm is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for every $x \in B(1, 1)$. And the p -adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for every $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.2. [36],[60] *Let $x \in B(0, p^{-1/(p-1)})$ then we have*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1, \quad |\log_p(1+x)|_p = |x|_p < p^{-1/(p-1)}$$

and

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1+x)) = 1+x.$$

In what follows, we will use the following auxiliary facts.

Lemma 2.3. [35] *If $|a_i|_p \leq 1$, $|b_i|_p \leq 1$, $i = 1, \dots, n$, then*

$$\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right|_p \leq \max_{1 \leq i \leq n} \{|a_i - b_i|_p\}$$

Put

$$\mathcal{E}_p = \{x \in \mathbb{Q}_p : |x|_p = 1, \quad |x - 1|_p < p^{-1/(p-1)}\}.$$

As corollary of Lemma 2.2 we have the following

Lemma 2.4. *The set \mathcal{E}_p has the following properties:*

- (a) \mathcal{E}_p is a group under multiplication;
- (b) $|a - b|_p < 1$ for all $a, b \in \mathcal{E}_p$;
- (c) If $a, b \in \mathcal{E}_p$ then it holds

$$|a + b|_p = \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ 1, & \text{if } p \neq 2. \end{cases}$$

- (d) If $a \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $a = \exp_p(h)$.

Note that the basics of p -adic analysis, p -adic mathematical physics are explained in [36, 51, 60].

2.2. p -adic measure. Let (X, \mathcal{B}) be a measurable space, where \mathcal{B} is an algebra of subsets X . A function $\mu : \mathcal{B} \rightarrow \mathbb{Q}_p$ is said to be a p -adic measure if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ ($i \neq j$) the equality holds

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

A p -adic measure is called a *probability measure* if $\mu(X) = 1$. One of the important condition (which was already invented in the first Monna–Springer theory of non-Archimedean integration [39]) is boundedness, namely a p -adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. We pay attention to an important special case in which boundedness condition by itself provides a fruitful integration theory (see for example [30]). Note that, in general, a p -adic probability measure need not be bounded [29, 31, 36]. For more detail information about p -adic measures we refer to [2, 29].

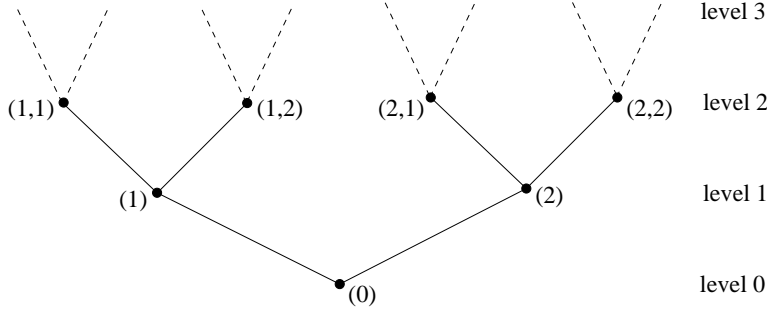


FIGURE 1. The first levels of Γ_+^2

2.3. Cayley tree. Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0 , which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y . The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y .

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^n W_m, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

The set of direct successors of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n.$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

Two vertices $x, y \in V$ are called the *next-nearest neighbors* if $d(x, y) = 2$. The next-nearest-neighbors vertices x and y are called the *prolonged next-nearest neighbors* if $x \in W_{n-2}$ and $y \in W_n$ for some $n \geq 1$, which are denoted by $\rangle x, y \langle$.

Now we are going to introduce a coordinate structure in Γ_+^k . Every vertex x (except for x^0) of Γ_+^k has coordinates (i_1, \dots, i_n) , here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0) (see Figure 1). Namely, the symbol (0) constitutes level 0 and the sites i_1, \dots, i_n form level n of the lattice. In this notation for $x \in \Gamma_+^k$, $x = \{i_1, \dots, i_n\}$ we have

$$S(x) = \{(x, i) : 1 \leq i \leq k\},$$

here (x, i) means that (i_1, \dots, i_n, i) .

Let us define on Γ_+^k a binary operation $\circ : \Gamma_+^k \times \Gamma_+^k \rightarrow \Gamma_+^k$ as follows, for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m)$$

and

$$y \circ x = (j_1, \dots, j_m) \circ (i_1, \dots, i_n) = (j_1, \dots, j_m, i_1, \dots, i_n).$$

By means of the defined operation Γ_+^k becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations $\tau_g : \Gamma_+^k \rightarrow \Gamma_+^k$, $g \in \Gamma_k$ by

$$\tau_g(x) = g \circ x.$$

Similarly, by means of τ_g one can define translation $\tilde{\tau}_g : L \rightarrow L$ of L . Namely,

$$\tilde{\tau}_g(\langle x, y \rangle) = \langle \tau_g(x), \tau_g(y) \rangle.$$

Let $G \subset \Gamma_+^k$ be a sub-semigroup of Γ_+^k and $h : L \rightarrow \mathbb{Q}_p$ be a function defined on L . We say that h is a G -periodic if $h(\tilde{\tau}_g(l)) = h(l)$ for all $g \in G$ and $l \in L$. Any Γ_+^k -periodic function is called *translation-invariant*. Put

$$G_m = \{x \in \Gamma_+^k : d(x, x^0) \equiv 0 \pmod{m}\}, \quad m \geq 2.$$

One can check that G_m is a sub-semigroup with a unit.

3. p -ADIC ISING-VANNIMENUS MODEL AND ITS p -ADIC GIBBS MEASURES

In this section we consider the p -adic Ising-Vannimenus model where spin takes values in the set $\Phi = \{-1, +1\}$, (Φ is called a *state space*) and is assigned to the vertices of the tree $\Gamma_+^k = (V, \Lambda)$. A configuration σ on V is then defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; in a similar manner one defines configurations σ and ω on V_n and W_n , respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma \vee \omega)(x) = \begin{cases} \sigma(x), & \text{if } x \in V_{n-1}, \\ \omega(x), & \text{if } x \in W_n. \end{cases}$$

It is clear that $\sigma \vee \omega \in \Omega_{V_n}$.

The Hamiltonian $H_n : \Omega_{V_n} \rightarrow \mathbb{Q}_p$ of the p -adic Ising-Vannimenus model has a form

$$(3.1) \quad H_n(\sigma) = J \sum_{\langle x, y \rangle \in L_n} \sigma(x)\sigma(y) + J_1 \sum_{\langle x, y \rangle : x \in V_n} \sigma(x)\sigma(y)$$

where $J_1, J \in B(0, p^{-1/(p-1)})$ are coupling constants.

Note that the last condition together with the strong triangle inequality implies the existence of $\exp_p(H_n(\sigma))$ for all $\sigma \in \Omega_{V_n}$, $n \in \mathbb{N}$. This is required to our construction.

Let $\mathbf{h} : \langle x, y \rangle \rightarrow \mathbf{h}_{xy} = (h_{xy,++}, h_{xy,+-}, h_{xy,-+}, h_{xy,--}) \in \mathbb{Q}_p^4$ be a vector valued function on L .

Given $n \in \mathbb{N}$, let us consider a p -adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} defined by

$$(3.2) \quad \mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp_p(H_n(\sigma)) \prod_{x \in W_{n-1}, y \in S(x)} (h_{xy, \sigma(x)\sigma(y)})^{\sigma(x)\sigma(y)}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor called a *partition function* given by

$$(3.3) \quad Z_n^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \exp_p(H_n(\sigma)) \prod_{x \in W_{n-1}, y \in S(x)} (h_{xy, \sigma(x)\sigma(y)})^{\sigma(x)\sigma(y)}.$$

Remark 3.1. We point out that, in general, in the definition of the measure (3.2), one can replace \exp_p by any p -adic number ρ . Such a kind of approach has been developed in [41, 42]. For the sake of simplicity, we restrict ourselves to the considered case. A more general case will be investigated elsewhere.

We recall [42, 53] that one of the central results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions, which is a well-known *Kolmogorov's extension Theorem* [56]. Therefore, in this paper we are interested in the same question but in a p -adic context. More exactly, we want to define a p -adic probability measure μ on Ω which is compatible with defined ones $\mu_{\mathbf{h}}^{(n)}$, i.e.

$$(3.4) \quad \mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \quad \text{for all } \sigma_n \in \Omega_{V_n}, n \in \mathbb{N}.$$

In general, à priori the existence such a kind of measure μ is not known, since there is not much information on topological properties, such as compactness, of the set of all p -adic measures defined even on compact spaces ². Note that certain properties of the set of p -adic measures has been studied in [22, 23], but those properties are not enough to prove the existence of the limiting measure. Therefore, at a moment, we can only use the p -adic Kolmogorov extension Theorem (see [17],[31]) which based on so called *compatibility condition* for the measures $\mu_{\mathbf{h}}^{(n)}$, $n \geq 1$, i.e.

$$(3.5) \quad \sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}),$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$. This condition according to the theorem implies the existence of a unique p -adic measure μ defined on Ω with a required condition (3.4). We should stress that using the compatibility condition for the Ising model on the Bethe lattice, in the real case, was started in [9] (see [53] for review). Note that more general theory of p -adic measures has been developed in [21].

Following [41, 42] if for some function \mathbf{h} the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition, then there is a unique p -adic probability measure, which we denote by $\mu_{\mathbf{h}}$. Such a measure $\mu_{\mathbf{h}}$ is said to be a *p -adic quasi Gibbs measure* corresponding to the p -adic Ising-Vannimenus model. By $Q\mathcal{G}(H)$ we denote the set of all p -adic quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_{xy}, \langle x, y \rangle \in L\}$. A p -adic quasi Gibbs measure defined by (3.2) is called *p -adic Gibbs measure* if $\mathbf{h}_{xy} \in \mathcal{E}_p^4$ for all $\langle x, y \rangle \in L$. The set of p -adic Gibbs measures we denote by $\mathcal{G}(H)$. If there are at least two distinct p -adic quasi Gibbs measures $\mu, \nu \in Q\mathcal{G}(H)$ such that μ is bounded and ν is unbounded, then we say that a *phase transition* occurs. By another words, one can find two different functions \mathbf{s} and \mathbf{h} defined on \mathbb{N} such that there exist the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$, for which one is bounded, another one is unbounded. Moreover, if there is a sequence of sets $\{A_n\}$ such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \rightarrow 0$ and $|\nu(A_n)|_p \rightarrow \infty$ as $n \rightarrow \infty$, then we say that there occurs a *strong phase transition*.

Remark 3.2. We would like to point out that if one considers the usual Ising model (i.e. in the real setting) on a multidimensional lattice, then at low temperature there occurs a phase transition, i.e. there exist q -different Gibbs measures μ_i , ($i = 1, \dots, q$), i.e.

$$\mu_+(\sigma(0) = 1) > 1/2, \quad \mu_-(\sigma(0) = -1) < 1/2 \quad j \neq i.$$

This implies that the measures μ_{\pm} are mutually singular to each other. The strong phase transition (see definition above), in the p -adic setting, has the similar meaning as singularity,

² In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e. Prohorov's Theorem). When the state space is non-compact, then there is a Dobrushin's Theorem [11, 12] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians.

i.e. the p -adic measures μ and ν are "singular" (in the above given sense). Here we have to stress that absolutely continuity and singularity of p -adic measures cannot be directly defined in a similar manner with real case. Absolutely continuity of p -adic measures have been studied in [21]. The singularity what we are proposing is consistent with that absolutely continuity introduced in [21].

Theorem 3.1. ([47]) *The measures $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \dots$ (see (3.2)) satisfy the compatibility condition (3.5) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$(3.6) \quad \begin{cases} h_{xy,++} \cdot h_{xy,-+} = \prod_{z \in S(y)} \frac{(ab)^2 h_{yz,++} h_{yz,-+} + 1}{a^2 h_{yz,++} h_{yz,-+} + b^2} \\ h_{xy,--} \cdot h_{xy,+-} = \prod_{z \in S(y)} \frac{(ab)^2 h_{yz,--} h_{yz,+-} + 1}{a^2 h_{yz,--} h_{yz,+-} + b^2} \\ h_{xy,++} \cdot h_{xy,+-} = \prod_{z \in S(y)} \frac{((ab)^2 h_{yz,++} h_{yz,+-} + 1) h_{yz,-+}}{(a^2 h_{yz,--} h_{yz,+-} + b^2) h_{yz,+-}} \end{cases}$$

where $a = \exp_p(J)$, $b = \exp_p(J_1)$.

Remark 3.3. If we take $J_1 = 0$, i.e. $b = 1$, then (3.6) reduces to the well-known equation for the Ising model (see [26, 46] for details). For this model the existence of two different p -adic quasi Gibbs measures has been considered in [24].

For convenience, let us denote

$$(3.7) \quad \begin{aligned} u_{xy,1} &= a^2 h_{xy,++} h_{xy,-+} \\ u_{xy,2} &= a^2 h_{xy,--} h_{xy,+-} \\ u_{xy,3} &= a^2 h_{xy,++} h_{xy,+-} \end{aligned}$$

and rewrite (3.6) by

$$(3.8) \quad \begin{cases} u_{xy,1} = a^2 \prod_{z \in S(y)} \frac{b^2 u_{yz,3} + 1}{u_{yz,3} + b^2} \\ u_{xy,2} = a^2 \prod_{z \in S(y)} \frac{(b^2 u_{yz,2} + 1) u_{yz,3}}{(u_{yz,3} + b^2) u_{yz,1}} \\ u_{xy,3} = a^2 \prod_{z \in S(y)} \frac{(b^2 u_{yz,3} + 1) u_{yz,1}}{(u_{yz,2} + b^2) u_{yz,3}} \end{cases}$$

Hence, due to Theorem 3.1 the problem of describing the p -adic quasi Gibbs measures is reduced to the description of solutions of the functional equations (3.8). It is worth mentioning that there are infinitely many solutions of the system of equations (3.6) corresponding to each solution of the system of equations (3.8). However, each solution of the system of equations (3.8) uniquely determines a p -adic quasi Gibbs measure.

Theorem 3.2. *There exists a unique p -adic quasi Gibbs measure $\mu_{\mathbf{u}}$ associated with the function $\mathbf{u} = \{\mathbf{u}_{xy}, \langle x, y \rangle \in L\}$ where $\mathbf{u}_{xy} = (u_{xy,1}, u_{xy,2}, u_{xy,3})$ is a solution of the system of equations (3.8). Moreover, the measure $\mu_{\mathbf{u}}$ is the p -adic Gibbs measure if and only if $\mathbf{u}_{xy} \in \mathcal{E}_p^3$ for all $\langle x, y \rangle \in L$.*

Proof. Let $\mathbf{u} = \{\mathbf{u}_{xy}, \langle x, y \rangle \in L\}$ be a function, where $\mathbf{u}_{xy} = (u_{xy,1}, u_{xy,2}, u_{xy,3})$ is a solution of the system of equations (3.8). Then, for any p -adic number $h_{xy,++} \in \mathbb{Q}_p \setminus \{0\}$, a function

$\mathbf{h} = \{\mathbf{h}_{xy}, \langle x, y \rangle \in L\}$ defined by

$$\mathbf{h}_{xy} = \left(h_{xy,++}, \frac{u_{xy,3}}{a^2 h_{xy,++}}, \frac{u_{xy,1}}{a^2 h_{xy,++}}, \frac{u_{xy,2} h_{xy,++}}{u_{xy,1}} \right)$$

is a solution of (3.6).

Now fix $n \geq 1$. Since $|W_{n-1}| = k^{n-1}$ and $|S(x)| = k$ we get $|L_n \setminus L_{n-1}| = k^n$. Let σ be any configuration in Ω_{V_n} . Denote

$$\begin{aligned} \mathcal{N}_{1,n}(\sigma) &= \{\langle x, y \rangle \in L_n \setminus L_{n-1} : \sigma(x) = 1, \sigma(y) = 1, x \in W_{n-1}, y \in S(x)\} \\ \mathcal{N}_{2,n}(\sigma) &= \{\langle x, y \rangle \in L_n \setminus L_{n-1} : \sigma(x) = 1, \sigma(y) = -1, x \in W_{n-1}, y \in S(x)\} \\ \mathcal{N}_{3,n}(\sigma) &= \{\langle x, y \rangle \in L_n \setminus L_{n-1} : \sigma(x) = -1, \sigma(y) = 1, x \in W_{n-1}, y \in S(x)\} \\ \mathcal{N}_{4,n}(\sigma) &= \{\langle x, y \rangle \in L_n \setminus L_{n-1} : \sigma(x) = -1, \sigma(y) = -1, x \in W_{n-1}, y \in S(x)\} \end{aligned}$$

We have

$$\begin{aligned} \prod_{\substack{x \in W_{n-1} \\ y \in S(x)}} (h_{xy, \sigma(x)\sigma(y)})^{\sigma(x)\sigma(y)} &= \prod_{\langle x, y \rangle \in \mathcal{N}_{1,n}(\sigma)} h_{xy,++} \prod_{\langle x, y \rangle \in \mathcal{N}_{2,n}(\sigma)} \frac{a^2 h_{xy,++}}{u_{xy,3}} \prod_{\langle x, y \rangle \in \mathcal{N}_{3,n}(\sigma)} \frac{a^2 h_{xy,++}}{u_{xy,1}} \\ &\quad \times \prod_{\langle x, y \rangle \in \mathcal{N}_{4,n}(\sigma)} \frac{u_{xy,2} h_{xy,++}}{u_{xy,1}} \\ &= \prod_{\langle x, y \rangle \in L_n \setminus L_{n-1}} h_{xy,++} \prod_{\langle x, y \rangle \in \mathcal{N}_{2,n}(\sigma)} \frac{a^2}{u_{xy,3}} \prod_{\langle x, y \rangle \in \mathcal{N}_{3,n}(\sigma)} \frac{a^2}{u_{xy,1}} \\ &\quad \times \prod_{\langle x, y \rangle \in \mathcal{N}_{4,n}(\sigma)} \frac{u_{xy,2}}{u_{xy,1}} \end{aligned}$$

By means of the last equalities, one can get from (3.2) and (3.3) that

$$\begin{aligned} \mu_h^{(n)}(\sigma) &= \frac{\exp_p(H_n(\sigma)) \prod_{\substack{x \in W_{n-1} \\ y \in S(x)}} (h_{xy, \sigma(x)\sigma(y)})^{\sigma(x)\sigma(y)}}{\sum_{\omega \in \Omega_{V_n}} \exp_p(H_n(\omega)) \prod_{\substack{x \in W_{n-1} \\ y \in S(x)}} (h_{xy, \omega(x)\omega(y)})^{\omega(x)\omega(y)}} \\ (3.9) \quad &= \frac{\exp_p(H_n(\sigma)) \prod_{\langle x, y \rangle \in \mathcal{N}_{2,n}(\sigma)} \frac{a^2}{u_{xy,3}} \prod_{\langle x, y \rangle \in \mathcal{N}_{3,n}(\sigma)} \frac{a^2}{u_{xy,1}} \prod_{\langle x, y \rangle \in \mathcal{N}_{4,n}(\sigma)} \frac{u_{xy,2}}{u_{xy,1}}}{\sum_{\omega \in \Omega_{V_n}} \exp_p(H_n(\omega)) \prod_{\langle x, y \rangle \in \mathcal{N}_{2,n}(\omega)} \frac{a^2}{u_{xy,3}} \prod_{\langle x, y \rangle \in \mathcal{N}_{3,n}(\omega)} \frac{a^2}{u_{xy,1}} \prod_{\langle x, y \rangle \in \mathcal{N}_{4,n}(\omega)} \frac{u_{xy,2}}{u_{xy,1}}} \end{aligned}$$

One can see the right hand side of (3.9) does not depend on $h_{xy,++}$. So, we can see that each solution \mathbf{u} of the system of equations (3.8) uniquely determines only one p -adic quasi Gibbs measure $\mu_{\mathbf{u}}$. \square

4. THE EXISTENCE OF p -ADIC QUASI GIBBS MEASURES

In this section we are going to establish the existence of p -adic quasi Gibbs measures by analyzing the equation (3.8).

Recall that a Γ_+^k -periodic function is called *translation-invariant*. Let $\mathbf{u} = \{\mathbf{u}_{xy}\}_{\langle x, y \rangle \in L}$ be a translation-invariant function i.e. $\mathbf{u}_{xy} = \mathbf{u}_{zw}$ for all $\langle x, y \rangle, \langle z, w \rangle \in L$. A p -adic quasi Gibbs measure $\mu_{\mathbf{u}}$, corresponding to a translation-invariant function \mathbf{u} , is called a *translation-invariant*

p-adic quasi Gibbs measure. The set of translation-invariant *p*-adic quasi Gibbs (resp. *p*-adic Gibbs) measures is denoted by $Q\mathcal{G}(H)_{TI}$ (resp. $\mathcal{G}(H)_{TI}$).

To solve the equation (3.8), in general, is very complicated. Therefore, let us first restrict ourselves to the description of translation-invariant solutions of (3.8). Therefore, (3.8) reduces to

$$(4.1) \quad \begin{cases} u_1 = a^2 \left(\frac{b^2 u_3 + 1}{u_3 + b^2} \right)^k \\ u_2 = a^2 \left(\frac{(b^2 u_2 + 1)u_3}{(u_3 + b^2)u_1} \right)^k \\ u_3 = a^2 \left(\frac{(b^2 u_3 + 1)u_1}{(u_2 + b^2)u_3} \right)^k \end{cases}$$

In what follows, we will always assume that $p \geq 3$ and $b \neq 1$.

4.1. The study of the system (4.1). In this subsection we are aiming to study the set of all solutions of the system (4.1).

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $F(\mathbf{u}) = (F_1(\mathbf{u}), F_2(\mathbf{u}), F_3(\mathbf{u}))$ be a mapping, where

$$F_1(\mathbf{u}) = a^2 \left(\frac{b^2 u_3 + 1}{u_3 + b^2} \right)^k, \quad F_2(\mathbf{u}) = a^2 \left(\frac{(b^2 u_2 + 1)u_3}{(u_3 + b^2)u_1} \right)^k, \quad F_3(\mathbf{u}) = a^2 \left(\frac{(b^2 u_3 + 1)u_1}{(u_2 + b^2)u_3} \right)^k.$$

Clearly that the solutions of (4.1) are fixed points of the mapping F . Therefore, the set $\mathbf{Fix}(F) = \{\mathbf{u} : F(\mathbf{u}) = \mathbf{u}\}$ describes all solutions of the system (4.1).

Proposition 4.1. *If $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{Fix}(F)$ then one has that $|u_2|_p = |u_3|_p = 1$.*

Proof. Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{Fix}(F)$ be a solution of (4.1).

Let us first prove that $|u_3|_p = 1$. Assume that $|u_3|_p \neq 1$. In this case, due to $|b^2 u_3 + 1|_p = |u_3 + b^2|_p$, from the first equation of (4.1) we obtain that $|u_1|_p = 1$. By substituting it, into the second and the third equations of (4.1), one finds

$$(4.2) \quad \begin{cases} |u_2|_p = |b^2 u_2 + 1|_p^k \cdot |u_3|_p^k \\ |u_3|_p = |u_2 + b^2|_p^{-k} \cdot |u_3|_p^{-k} \end{cases} \quad \text{whenever } |u_3|_p < 1$$

and

$$(4.3) \quad \begin{cases} |u_2|_p = |b^2 u_2 + 1|_p^k \\ |u_3|_p = |u_2 + b^2|_p^{-k} \end{cases} \quad \text{whenever } |u_3|_p > 1$$

It follows from the first equality of (4.2) that $|u_2|_p \neq 1$. Then, by multiplying both equalities of (4.2), one gets that $|u_2 u_3|_p = 1$. Therefore, one has that $|u_2|_p = |u_3|_p^{-1} > 1$. However, by substituting the last one into the second equality of (4.2), we have that $|u_3|_p = 1$ which is a contradiction.

Similarly, from the first equality of (4.3), we immediately get that $|u_2|_p = |b^2 u_2 + 1|_p = 1$. It then follows that $|u_2 + b^2|_p = |b^2 u_2 + b^4|_p = |(b^2 u_2 + 1) + (b^4 - 1)|_p = |b^2 u_2 + 1|_p = 1$. However, by substituting the last equality into the second equation of (4.3), one gets that $|u_3|_p = 1$ which is again a contradiction. These contradictions imply that $|u_3|_p = 1$.

Now, we are going to show that $|u_2|_p = 1$. Again assume the contrary, i.e. $|u_2|_p \neq 1$. In this case, we have $|b^2 u_2 + 1|_p = |u_2 + b^2|_p$. By multiplying the second and the third equations of (4.1), one finds $\frac{u_2 u_3}{u_1} = a^2 \left(\frac{b^2 u_2 + 1}{u_2 + b^2} \right)^k$. The last equality with $|u_3|_p = 1$ yields that $|u_1|_p = |u_2|_p \neq 1$. It

then follows from the first equation of (4.1) that $|u_1|_p = \frac{|b^2u_3+1|_p}{|u_3+b^2|_p} \neq 1$. On the other hand, one has that $|b^2u_3+1|_p = 1$ if and only if $|u_3+b^2|_p = 1$. Therefore, it implies that $|b^2u_3+1|_p < 1$. From this inequality together with the third equation of (4.1) one finds that $|u_3|_p < \left(\frac{|u_1|_p}{|u_2+b^2|_p}\right)^k$. Since $|u_1|_p = |u_2|_p \neq 1$, one has that $\left(\frac{|u_1|_p}{|u_2+b^2|_p}\right)^k \leq 1$. Hence, we obtain that $|u_3|_p < 1$ which is a contradiction. This completes the proof. \square

Proposition 4.2. *The following statements hold:*

- (i) *One has that $\mathbf{Fix}(F) \subset (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) \cup (\mathbb{Q}_p \times (-\mathcal{E}_p) \times (-\mathcal{E}_p))$;*
- (ii) *If $\mathbf{Fix}(F) \setminus (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) \neq \emptyset$ then -1 is the k^{th} residue of module p .*

Proof. (i) Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{Fix}(F)$ be a solution of the system (4.1).

We first show that if $u_3 \notin -\mathcal{E}_p$ then $\mathbf{u} \in \mathcal{E}_p^3$. Indeed, due to Proposition 4.1, we have that $|u_2|_p = |u_3|_p = 1$, $|u_3+1|_p = 1$, and

$$\frac{b^2u_3+1}{u_3+b^2} = \frac{b^2 + \frac{1-b^2}{u_3+1}}{1 + \frac{b^2-1}{u_3+1}} \in \mathcal{E}_p.$$

By using these, from the first equation of (4.1) one finds that $u_1 \in \mathcal{E}_p$. Due to $|u_1|_p = |u_2|_p = |u_3|_p = 1$ together with the second equation of (4.1) we get that $|b^2u_2+1|_p = 1$, i.e. $u_2 \notin -\mathcal{E}_p$. From

$$u_2u_3 = a^2u_1 \left(\frac{b^2u_2+1}{u_2+b^2} \right)^k, \quad \frac{b^2u_2+1}{u_2+b^2} = \frac{b^2 + \frac{1-b^2}{u_2+1}}{1 + \frac{b^2-1}{u_2+1}} \in \mathcal{E}_p$$

it follows that $u_2 \cdot u_3 \in \mathcal{E}_p$. This means that

$$\frac{b^2u_3+1}{(u_2+b^2)u_3} = \frac{b^2u_3+1}{u_2u_3+b^2u_3} = \frac{1}{1 + \frac{u_2u_3-1}{b^2u_3+1}} \in \mathcal{E}_p.$$

Hence, from the third equation of (4.1) one gets $u_3 \in \mathcal{E}_p$, and consequently, $u_2 \in \mathcal{E}_p$.

Now, we want to show that if $u_3 \in -\mathcal{E}_p$ then $u_2 \in -\mathcal{E}_p$. We assume the contrary, i.e. $u_2 \notin -\mathcal{E}_p$. From $|u_2|_p = |u_3|_p = 1$ and

$$u_2u_3 = a^2u_1 \left(\frac{b^2u_2+1}{u_2+b^2} \right)^k, \quad \frac{b^2u_2+1}{u_2+b^2} = \frac{b^2 + \frac{1-b^2}{u_2+1}}{1 + \frac{b^2-1}{u_2+1}} \in \mathcal{E}_p$$

we get that $|u_1|_p = 1$. By substituting it, into the third equation of (4.1), one has that $|b^2u_3+1|_p = 1$ which contradicts to $u_3 \in -\mathcal{E}_p$. This means that $u_2 \in -\mathcal{E}_p$.

(ii) Assume that a solution $\mathbf{u} = (u_1, u_2, u_3)$ of the system (4.1) does not belong to \mathcal{E}_p^3 . Then we want to show that -1 is the k^{th} residue of module p . Again we suppose the contrary, i.e. $\alpha^k \not\equiv -1 \pmod{p}$ for all $\alpha \in \{1, 2, \dots, p-1\}$. Due to Proposition 4.1, we have that $|u_2|_p = |u_3|_p = 1$. Let $y = \frac{u_3}{a^2}$ and $z = \frac{(b^2u_3+1)u_1}{(u_2+b^2)u_3}$. It is clear that $|y|_p = 1$ and it follows from the third equation of (4.1) that $z^k = y$, so, $|z|_p = 1$. Therefore, one has $z = z_0 + z_1p + z_2p^2 + \dots$ and $y = y_0 + y_1p + y_2p^2 + \dots$ such that $y_0 \equiv z_0^k \not\equiv -1 \pmod{p}$. Consequently, from $u_3 = a^2y$ we obtain that $u_3 \notin -\mathcal{E}_p$. This, according to the previous case, yields that $\mathbf{u} \in \mathcal{E}_p^3$ which is a contradiction. This completes the proof. \square

Remark 4.1. According to Theorem 3.2, if -1 is not the k^{th} residue of module p then any translation-invariant p -adic quasi Gibbs measure is a p -adic Gibbs measure.

Our next aim is to show that $\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) \neq \emptyset$. For that purpose, we are searching for a solution of the system of equations (4.1) in the form $u_1 = u_2 = u_3 = u$. In that case, the system (4.1) reduces to

$$(4.4) \quad u = a^2 \left(\frac{b^2 u + 1}{u + b^2} \right)^k.$$

Let us consider a function $f_{a,b,k} : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ defined by

$$(4.5) \quad f_{a,b,k}(u) = a^2 \left(\frac{b^2 u + 1}{u + b^2} \right)^k, \quad b \neq 1$$

Note that in the real setting, the fixed points and dynamics of the function f defined by (4.5) were studied very well in the classical textbooks of statistical mechanics (for the latest book see [53]). Such a function (4.5) is called *the Ising-Potts mapping*. The Ising-Potts mapping may exhibit a chaotic behavior (see [10], [40]). Here, in the p -adic field, we would like to examine the set $\mathbf{Fix}(f_{a,b,k}) = \{u \in \mathbb{Q}_p : f_{a,b,k}(u) = u\}$ of fixed points of the function (4.5). The dynamics of the function $f_{a,b,k}$ over the p -adic field will be studied elsewhere.

Theorem 4.3. *Let $p \geq 3$, $a, b \in \mathcal{E}_p$, $b \neq 1$ and $f_{a,b,k} : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ be a function defined by (4.5). Then the following statements hold:*

- (i) *One has that $\mathbf{Fix}(f_{a,b,k}) \subset \mathcal{E}_p \cup (-\mathcal{E}_p)$ and $|\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p| = 1$;*
- (ii) *If $\frac{p-1}{(k,p-1)}$ is odd then $\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p) = \emptyset$;*
- (iii) *If $|k|_p = 1$ and $\frac{p-1}{(k,p-1)}$ is even then $|\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p)| = (k, p-1)$.*
- (iv) *If k is odd and $\max\{|k|_p, |a-1|_p\} < |b-1|_p$ then $|\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p)| \geq 1$.*

Proof. (i) Due to Proposition 4.1, one has that $\mathbf{Fix}(f_{a,b,k}) \subset \mathcal{E}_p \cup (-\mathcal{E}_p)$. Let us show that $|\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p| = 1$. Putting $y = \frac{u}{a^2}$ from (4.4) one gets that

$$(4.6) \quad y = \left(\frac{a^2 b^2 y + 1}{a^2 y + b^2} \right)^k.$$

This yields that any solution y of the equation (4.6) is the k^{th} power of some p -adic number $x \in \mathbb{Q}_p$, i.e. $y = x^k$. It follows from (4.6) that

$$(4.7) \quad x^k = \left(\frac{a^2 b^2 x^k + 1}{a^2 x^k + b^2} \right)^k \quad \text{or} \quad \left(\frac{a^2 b^2 x^k + 1}{a^2 x^{k+1} + x b^2} \right)^k = 1$$

Let $\eta = \frac{a^2 b^2 x^k + 1}{a^2 x^{k+1} + x b^2}$, then η is a solution of the equation $\eta^k = 1$. The last equation has the following solutions $\eta_1 (= 1), \eta_2, \dots, \eta_d$ in \mathbb{Q}_p , where $d = (k, p-1)$. Therefore, we have

$$(4.8) \quad x = \eta_i \frac{a^2 b^2 x^k + 1}{a^2 x^k + b^2}, \quad i = \overline{1, d}$$

As for two different $i_1 \neq i_2$, the corresponding solutions x_{i_1}, x_{i_2} of the equation (4.8) satisfy the condition $\frac{x_{i_1}}{\eta_{i_1}} = \frac{x_{i_2}}{\eta_{i_2}}$. Therefore, it is enough to study the equation (4.8) for $\eta_1 = 1$.

Let $g_{a,b,k} : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ be a function given by

$$(4.9) \quad g_{a,b,k}(x) = \frac{a^2 b^2 x^k + 1}{a^2 x^k + b^2}.$$

Consequently, we conclude that

$$(4.10) \quad \mathbf{Fix}(f_{a,b,k}) = \{u : u = a^2 x^k, x \in \text{Fix}(g_{a,b,k})\}.$$

Due to $\mathbf{Fix}(f_{a,b,k}) \subset \mathcal{E}_p \cup (-\mathcal{E}_p)$, it follows from (4.10) that

$$\mathbf{Fix}(g_{a,b,k}) \subset \{x \in \mathbb{Z}_p^* : x_0^k \equiv 1 \pmod{p}\} \cup \{x \in \mathbb{Z}_p^* : x_0^k \equiv -1 \pmod{p}\}.$$

Let $x \in \mathbf{Fix}(g_{a,b,k}) \cap \mathbb{Z}_p^*$ such that $x_0^k \equiv 1 \pmod{p}$. We then get that $|x^k + 1|_p = 1$. This implies that

$$(4.11) \quad x = \frac{a^2 b^2 x^k + 1}{a^2 x^k + b^2} = b^2 \frac{1 - \frac{a^2 b^2 - 1}{a^2 b^2 (x^k + 1)}}{1 + \frac{b^2 - a^2}{a^2 (x^k + 1)}} \in \mathcal{E}_p.$$

Thus, we obtain

$$(4.12) \quad \mathbf{Fix}(g_{a,b,k}) \subset \mathcal{E}_p \cup \{x \in \mathbb{Z}_p^* : x_0^k \equiv -1 \pmod{p}\}.$$

It is clear that $g_{a,b,k}(\mathcal{E}_p) \subset \mathcal{E}_p$. For any $x, y \in \mathcal{E}_p$, we get that

$$(4.13) \quad g_{a,b,k}(x) - g_{a,b,k}(y) = \frac{a^2(b^4 - 1) \sum_{i=0}^{k-1} x^{k-1-i} y^i}{(a^2 x^k + b^2)(a^2 y^k + b^2)}(x - y).$$

According to Lemma 2.4 we have that

$$(4.14) \quad \begin{cases} |b^4 - 1|_p \leq \frac{1}{p}, & \left| \sum_{i=0}^{k-1} x^{k-1-i} y^i \right|_p \leq 1, \\ |a^2 x^k + b^2|_p = 1, & |a^2 y^k + b^2|_p = 1. \end{cases}$$

By using (4.14) and (4.13), we obtain that

$$|g_{a,b,k}(x) - g_{a,b,k}(y)|_p \leq \frac{1}{p} |x - y|_p.$$

This means that $g_{a,b,k}$ is a contraction on \mathcal{E}_p . Since \mathcal{E}_p is compact, the function $g_{a,b,k}$ has a unique fixed point in \mathcal{E}_p . Hence, due to (4.10), we have that $|\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p| = 1$.

(ii) We know that $\sqrt[k]{-1}$ does not exist in \mathbb{F}_p if and only if $\frac{p-1}{(k,p-1)}$ is odd. In this case, due to (4.10) and (4.12), we have that $\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p) = \emptyset$.

(iii) Let $(k, p) = 1$ and $\frac{p-1}{(k,p-1)}$ is even. Let us establish that $|\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p)| = (k, p-1)$. Under our assumption, $\sqrt[k]{-1}$ exists in \mathbb{F}_p . Let α be some solution of the congruent equation $\alpha^k + 1 \equiv 0 \pmod{p}$. From $g_{a,b,k}(x) = x$ one finds that

$$(4.15) \quad \mathbf{g}(x) \equiv a^2 x^{k+1} - a^2 b^2 x^k + b^2 x - 1 = 0.$$

Since $\alpha \not\equiv 1 \pmod{p}$ and $(k, p) = 1$, we then have that

$$\begin{aligned} \mathbf{g}(\alpha) &= -a^2 \alpha + a^2 b^2 + b^2 \alpha - 1 = \alpha(b^2 - a^2) + a^2 b^2 - 1 \equiv 0 \pmod{p}, \\ \alpha \mathbf{g}'(\alpha) &= -(k+1)a^2 \alpha + k a^2 b^2 + b^2 \alpha = \alpha(b^2 - a^2) + k a^2 (b^2 - \alpha) \not\equiv 0 \pmod{p}. \end{aligned}$$

According to the Hensel's lemma there exists a unique p -adic integer x_α such that

$$\mathbf{g}(x_\alpha) = 0, \quad x_\alpha \equiv \alpha \pmod{p}.$$

Consequently, for each solution of the congruent equation $\alpha^k + 1 \equiv 0 \pmod{p}$, there exists a unique solution x_α of $g_{a,b,k}(x) = x$ such that $x_\alpha \equiv \alpha \pmod{p}$. Hence $|\mathbf{Fix}(g_{a,b,k}) \cap (-\mathcal{E}_p)| = (k, p-1)$ or equivalently $|\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p)| = (k, p-1)$.

(iv) Let k be odd and $r = |b - 1|_p$. We then have that $f_{a,b,k}(-1) = -a^2$ and

$$(4.16) \quad f_{a,b,k}(u) - f_{a,b,k}(v) = \frac{a^2(b^4 - 1) \sum_{i=0}^{k-1} [(b^2u + 1)(v + b^2)]^{k-1-i} [(b^2v + 1)(u + b^2)]^i}{(u + b^2)^k(v + b^2)^k} (u - v).$$

for all $u, v \in B(-1, r)$. From $|u + 1|_p < |b - 1|_p$ and $|v + 1|_p < |b - 1|_p$ one has

$$(4.17) \quad \begin{cases} |b^2u + 1|_p = |b^2v + 1|_p = |u + b^2|_p = |v + b^2|_p = |b - 1|_p, \\ \left| \sum_{i=0}^{k-1} [(b^2u + 1)(v + b^2)]^{k-1-i} [(b^2v + 1)(u + b^2)]^i \right|_p \leq |k(b - 1)^{2k-2}|_p. \end{cases}$$

By plugging (4.17) into (4.16), we obtain

$$(4.18) \quad |f_{a,b,k}(u) - f_{a,b,k}(v)|_p \leq \frac{|k|_p}{|b - 1|_p} \cdot |u - v|_p.$$

Since $\max\{|k|_p, |a - 1|_p\} < |b - 1|_p$, one has that $-a^2 \in B(-1, r)$ and $f_{a,b,k}(B(-1, r)) \subset B(-a^2, r_1) \subset B(-1, r)$ with $r_1 < r$. Thus, the function $f_{a,b,k}$ is a contraction on $B(-1, r)$. Consequently, there exists a unique fixed point in $B(-1, r) \subset (-\mathcal{E}_p)$, i.e. $|\mathbf{Fix}(f_{a,b,k}) \cap (-\mathcal{E}_p)| \geq 1$. \square

Remark 4.2. We stress that the equality $|\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p| = 1$ can be directly proven by means of the results of [45].

The following result immediately follows from Theorem 4.3.

Corollary 4.4. *Let $p \geq 3$ and $a, b \in \mathcal{E}_p$, $b \neq 1$. The following statements hold true:*

- (i) *One has that $|\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)| \geq 1$;*
- (ii) *If $\frac{p-1}{(k,p-1)}$ is odd then $\mathbf{Fix}(F) \cap (\mathbb{Q}_p \times (-\mathcal{E}_p) \times (-\mathcal{E}_p)) = \emptyset$;*
- (iii) *If $|k|_p = 1$ and $\frac{p-1}{(k,p-1)}$ is even then $|\mathbf{Fix}(F) \cap ((-\mathcal{E}_p) \times (-\mathcal{E}_p) \times (-\mathcal{E}_p))| \geq (k, p - 1)$.*
- (iv) *If k is odd and $\max\{|k|_p, |a - 1|_p\} < |b - 1|_p$ then $|\mathbf{Fix}(F) \cap ((-\mathcal{E}_p) \times (-\mathcal{E}_p) \times (-\mathcal{E}_p))| \geq 1$.*

The following theorem gives the precise description of the set $\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)$.

Theorem 4.5. *Let $p \geq 3$ and $a, b \in \mathcal{E}_p$, $b \neq 1$. One has that $|\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)| = 1$ and*

$$\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) = (\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p) \times (\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p) \times (\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p).$$

Proof. In order to describe the set $\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)$, we have to solve the system of equation (4.1) in the set $\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p$. Let us define the polynomial function $G : \mathbb{Q}_p^3 \rightarrow \mathbb{Q}_p^3$ $G(\mathbf{u}) = (G_1(\mathbf{u}), G_2(\mathbf{u}), G_3(\mathbf{u}))$ with p -adic integer coefficients as follows

$$\begin{aligned} G_1(\mathbf{u}) &= u_1(u_3 + b^2)^k - a^2(b^2u_3 + 1)^k, \\ G_2(\mathbf{u}) &= u_1^k u_2(u_3 + b^2)^k - a^2(b^2u_2 + 1)^k u_3^k, \\ G_3(\mathbf{u}) &= (u_2 + b^2)^k u_3^{k+1} - a^2 u_1^k (b^2u_3 + 1)^k. \end{aligned}$$

It is clear that $\mathbf{N}_G(\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) = \mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)$ where $\mathbf{N}_G(\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) = \{\mathbf{u} \in \mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p : G(\mathbf{u}) = \Theta\}$. So, we apply the generalized Hensel's lemma in order to describe the set $\mathbf{N}_G(\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)$.

It is easy to check that $G(1, 1, 1) \equiv \Theta \pmod{p}$ and $\det(\mathbb{J}(G(1, 1, 1))) \equiv 2^{3k} \not\equiv 0 \pmod{p}$, where

$$\mathbb{J}(G(1, 1, 1)) \equiv \begin{pmatrix} 2^k & 0 & 0 \\ 2^k k & 2^{k-1}(2-k) & -2^{k-1}k \\ -2^k k & 2^{k-1}k & 2^{k-1}(k+2) \end{pmatrix} \pmod{p}$$

Thus, due to the generalized Hensel's Lemma 2.1 there exist a unique root \mathbf{u}_* of the polynomial equation $G(\mathbf{u}) = \Theta$ which satisfies the condition $\mathbf{u}_* \equiv (1, 1, 1) \pmod{p}$. Consequently, $|\mathbf{N}_G(\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)| = |\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p)| = 1$.

On the other hand, we already knew that the system of equation (4.1) has a solution of the form $\mathbf{u} = (u, u, u)$ in the set $\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p$, where u is a unique fixed of the function f given by (4.5) in the set \mathcal{E}_p . Therefore, we get that $\mathbf{Fix}(F) \cap (\mathcal{E}_p \times \mathcal{E}_p \times \mathcal{E}_p) = (\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p) \times (\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p) \times (\mathbf{Fix}(f_{a,b,k}) \cap \mathcal{E}_p)$. \square

4.2. The existence of translation-invariant p -adic quasi Gibbs measures. By means of Theorem 3.2 and Corollary 4.4, we may construct translation invariant p -adic quasi Gibbs measures.

Theorem 4.6. *Let $p \geq 3$ and $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. The following assertions hold true for the p -adic Ising-Vannimenus model (3.1) on a Cayley tree of any order k :*

- (i) *There always exists at least one translation-invariant p -adic Gibbs measure, i.e. $|\mathcal{G}(H)_{TI}| \geq 1$;*
- (ii) *If $\frac{p-1}{(k,p-1)}$ is odd then there is no translation-invariant p -adic quasi Gibbs measure, i.e. $Q\mathcal{G}(H)_{TI} \setminus \mathcal{G}(H)_{TI} = \emptyset$;*
- (iii) *If $|k|_p = 1$ and $\frac{p-1}{(k,p-1)}$ is even then there are at least $(k, p-1)$ - number of translation-invariant p -adic quasi Gibbs measures, i.e. $|Q\mathcal{G}(H)_{TI} \setminus \mathcal{G}(H)_{TI}| \geq (k, p-1)$;*
- (iv) *If k is odd and $\max\{|k|_p, |a-1|_p\} < |b-1|_p$ then there is at least one translation-invariant p -adic quasi Gibbs measure, i.e. $|Q\mathcal{G}(H)_{TI} \setminus \mathcal{G}(H)_{TI}| \geq 1$*

4.3. The existence of periodic p -adic quasi Gibbs measures. All previous results were concerned with translation-invariant p -adic (quasi) Gibbs measures. In this section, we are aiming to show an existence of periodic (non-translation invariant) p -adic Gibbs measures for the p -adic Ising-Vannimenus model on a Cayley tree of order two. Recall that $G_2 = \{x \in \Gamma_+^2 : d(x, x^0) \equiv 0 \pmod{2}\}$ is a sub-semigroup of Γ_+^2 . The function $\mathbf{u} = \{\mathbf{u}_{xy}\}_{\langle x,y \rangle \in L}$ is called two periodic if $\mathbf{u}(\tilde{\tau}_g(l)) = \mathbf{u}(l), \forall g \in G_2, l \in L$.

Theorem 4.7. *Let $p \equiv 1 \pmod{4}$ and $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. Then there exist 2-periodic p -adic quasi Gibbs measures for the p -adic Ising-Vannimenus model on a Cayley tree of order two.*

Proof. In order to describe the 2-periodic p -adic quasi Gibbs measures, we have to solve the following system of equations

$$(4.19) \quad \begin{cases} u_1 = a^2 \left(\frac{b^2 v_3 + 1}{v_3 + b^2} \right)^2, & v_1 = a^2 \left(\frac{b^2 u_3 + 1}{u_3 + b^2} \right)^2, \\ u_2 = a^2 \left(\frac{(b^2 v_2 + 1)v_3}{(v_3 + b^2)v_1} \right)^2, & v_2 = a^2 \left(\frac{(b^2 u_2 + 1)u_3}{(u_3 + b^2)u_1} \right)^2, \\ u_3 = a^2 \left(\frac{(b^2 v_3 + 1)v_1}{(v_2 + b^2)v_3} \right)^2, & v_3 = a^2 \left(\frac{(b^2 u_3 + 1)u_1}{(u_2 + b^2)u_3} \right)^2. \end{cases}$$

For sake of the simplicity, we are looking for a solution of the form $\mathbf{u} = (u, u, u)$. We then obtain from (4.19) that

$$f_{a,b,2}(f_{a,b,2}(u)) = u.$$

It is clear that any solution u of the equation $f_{a,b,2}(f_{a,b,2}(u)) = u$ should be a prefect square of some p -adic number x , i.e., $u = x^2$. So, we have to solve the following equation with respect to x

$$g_{a,b,2}(g_{a,b,2}(x)) = x.$$

Note that any solution of the equation $g_{a,b,2}(u) = u$ is also solution of the equation $g_{a,b,2}(g_{a,b,2}(u)) = u$. However, we are looking for only 2-period solutions. Hence, we have to solve the equation $\frac{g_{a,b,2}(g_{a,b,2}(u)) - u}{g_{a,b,2}(u) - u} = 0$. This is equivalent to the following equation

$$(4.20) \quad b^2(a^2b^2 + 1)x^2 + a(b^4 - 1)x + b^2(b^2 + a^2) = 0.$$

The discriminant of the quadratic equation (4.20) is

$$\Delta(a, b) = a^2(b^4 - 1)^2 - 4b^4(a^2b^2 + 1)(b^2 + a^2).$$

Since $a \equiv 1 \pmod{p}$ and $b \equiv 1 \pmod{p}$, we get that

$$(4.21) \quad \Delta(1, 1) \equiv -16 \pmod{p}.$$

Thus, $\sqrt{\Delta(a, b)}$ exists in \mathbb{Q}_p if and only if $\sqrt{-1}$ exists in \mathbb{Q}_p or equivalently $p \equiv 1 \pmod{4}$. In this case, the quadratic equation (4.20) has two solutions

$$x_{\pm} = \frac{-a(b^4 - 1) \pm \sqrt{\Delta(a, b)}}{2b^2(a^2b^2 + 1)}.$$

Due to Theorem 3.2, there exists a 2-periodic p -adic quasi Gibbs measures associated with $\mathbf{u}_{\pm} = (x_{\pm}^2, x_{\pm}^2, x_{\pm}^2)$. This completes the proof. \square

5. UNIQUENESS OF p -ADIC GIBBS MEASURE

In the previous section we have shown that if $p \geq 3$, then the set $\mathcal{G}(H)$ is not empty for the p -adic Ising-Vannimenus model on a Cayley tree of order $k \geq 1$. In this section we are going to study the uniqueness of p -adic Gibbs measure for this model.

Theorem 5.1. *Let $p \geq 3$ and $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. Assume that $\mu_{\mathbf{u}}$ be a p -adic Gibbs measure associated with the function $\mathbf{u} = \{\mathbf{u}_{xy}\}_{\langle x, y \rangle \in L}$. Then $\mu_{\mathbf{u}}$ is a translation-invariant if one of the following conditions holds:*

- (i) $u_{xy,1} = u_{xy,2}$ for all $\langle x, y \rangle \in L$;
- (ii) $u_{xy,2} = u_{xy,3}$ for all $\langle x, y \rangle \in L$;
- (iii) $u_{xy,1} = u_{xy,3}$ for all $\langle x, y \rangle \in L$.

Proof. Due to Theorem 3.2, it is enough to show that the system of equations (3.8) has a unique solution whenever either one of the conditions (i) – (iii) is satisfied. It follows from (3.8) that

$$(5.1) \quad \begin{cases} u_{xy,1} = a^2 \prod_{z \in S(y)} \frac{b^2 u_{yz,3} + 1}{u_{yz,3} + b^2} \\ u_{xy,3} = a^2 \prod_{z \in S(y)} \frac{b^2 u_{yz,1} + 1}{u_{yz,1} + b^2} \end{cases} \quad \text{if} \quad u_{xy,1} = u_{xy,2} \quad \forall \langle x, y \rangle \in L$$

$$(5.2) \quad \begin{cases} u_{xy,1} = a^2 \prod_{z \in S(y)} \frac{b^2 u_{yz,1} + 1}{u_{yz,1} + b^2} \\ u_{xy,2} = a^2 \prod_{z \in S(y)} \frac{b^2 u_{yz,2} + 1}{u_{yz,2} + b^2} \end{cases} \quad \text{if} \quad u_{xy,1} = u_{xy,3} \quad \forall \langle x, y \rangle \in L$$

$$(5.3) \quad \begin{cases} u_{xy,1} = a^2 \prod_{z \in S(y)} \frac{b^2 u_{yz,1} + 1}{u_{yz,1} + b^2} \\ u_{xy,1} = u_{xy,2} = u_{xy,3} \end{cases} \quad \text{if} \quad u_{xy,2} = u_{xy,3} \quad \forall \langle x, y \rangle \in L$$

Let us consider the case (i), i.e., $u_{xy,1} = u_{xy,2}$ for all $\langle x, y \rangle \in L$. We can apply the same method for the rest cases.

The main idea is to show that the mapping $G : \mathcal{E}_p^2 \rightarrow \mathcal{E}_p^2$, $G = (G_1, G_2)$ defined as below is contraction with respect to the p -adic norm $\|\mathbf{u}\|_p = \max\{|u_1|_p, |u_2|_p\}$ in the set \mathcal{E}_p^2

$$G_1(\mathbf{u}) = \frac{b^2 u_2 + a^{-2}}{u_2 + a^{-2} b^2}, \quad G_2(\mathbf{u}) = \frac{b^2 u_1 + a^{-2}}{u_1 + a^{-2} b^2}, \quad \mathbf{u} = (u_1, u_2) \in \mathcal{E}_p^2.$$

One has for any $\mathbf{u}, \mathbf{v} \in \mathcal{E}_p^2$ that

$$(5.4) \quad |G_i(\mathbf{u}) - G_i(\mathbf{v})|_p = \frac{|(b^4 - 1)(u_j - v_j)|_p}{|a^2(u_j + a^{-2}b^2)(v_j + a^{-2}b^2)|_p} = |b - 1|_p \cdot |u_j - v_j|_p \leq \frac{1}{p} \|\mathbf{u} - \mathbf{v}\|_p$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Therefore, $\|G(\mathbf{u}) - G(\mathbf{v})\|_p \leq \|\mathbf{u} - \mathbf{v}\|_p$, i.e., $G : \mathcal{E}_p^2 \rightarrow \mathcal{E}_p^2$ is a contraction mapping.

Let $\{\mathbf{u}^{(r)}\}_{r \in \mathbb{N}}$ and $\{\mathbf{v}^{(r)}\}_{r \in \mathbb{N}}$ be sequences in \mathcal{E}_p^2 . For any $n \geq 1$, we have that

$$\prod_{s=1}^n G_i(\mathbf{u}^{(s)}) - \prod_{s=1}^n G_i(\mathbf{v}^{(s)}) = \sum_{l > s} \left(\prod_{l > s} G_i(\mathbf{u}^{(l)}) (G_i(\mathbf{u}^{(s)}) - G_i(\mathbf{v}^{(s)})) \prod_{m < s} G_i(\mathbf{v}^{(m)}) \right)$$

By means of Lemma 2.4 and (5.4), we can find that

$$(5.5) \quad \left| \prod_{s=1}^n G_i(\mathbf{u}^{(s)}) - \prod_{s=1}^n G_i(\mathbf{v}^{(s)}) \right|_p \leq \frac{1}{p} \max_{1 \leq s \leq n} \{\|\mathbf{u}^{(s)} - \mathbf{v}^{(s)}\|_p\}.$$

We define the mapping $\mathcal{F} : \mathcal{E}_p^2 \rightarrow \mathcal{E}_p^2$, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ as follows

$$\mathcal{F}_1(\mathbf{u}_{xy}) = \prod_{z \in S(y)} G_1(\mathbf{u}_{yz}), \quad \mathcal{F}_2(\mathbf{u}_{xy}) = \prod_{z \in S(y)} G_2(\mathbf{u}_{yz}), \quad \langle x, y \rangle \in L$$

Then by means of (5.5), one has for all $\langle x, y \rangle \in L$ that

$$(5.6) \quad \|\mathcal{F}(\mathbf{u}_{xy}) - \mathcal{F}(\mathbf{v}_{xy})\|_p \leq \frac{1}{p} \max_{z \in S(y)} \|\mathbf{u}_{yz} - \mathbf{v}_{yz}\|_p.$$

Let $\mathbf{u} = \{\mathbf{u}_{xy}\}$ and $\mathbf{v} = \{\mathbf{v}_{xy}\}$ be solutions of (5.1). Then for any $n \geq 1$ from (5.6) one gets

$$\|\mathcal{F}(\mathbf{u}_{xy}) - \mathcal{F}(\mathbf{v}_{xy})\|_p \leq \frac{1}{p^n}, \quad \text{for all } \langle x, y \rangle \in L$$

Since arbitrarily of n we obtain $\mathbf{u} = \mathbf{v}$. Thus, we have shown that (5.1) has no more one solution. If $u_{xy,i} = u_i$, $i = 1, 3$ for all $\langle x, y \rangle \in L$ we get $\mathcal{F} \equiv F$. From compactness of \mathcal{E}_p^2 the function F (respectively function \mathcal{F}) has a unique fixed point on \mathcal{E}_p^2 . This means that $\mu_{\mathbf{u}}$ is a translation-invariant. \square

From this theorem and Theorem 5.2 we immediately obtain the following

Corollary 5.2. *Let $p \geq 3$ and $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. Then there exists a unique translation-invariant p -adic Gibbs measure corresponding to the model (3.1) on the Cayley tree of order $k \geq 1$.*

Remark 5.1. The proved Theorem 5.1 and Corollary 5.2 partially confirm the conjecture formulated in [47].

Remark 5.2. Note that the uniqueness of p -adic Gibbs measures for the general nearest neighbor interaction models (which are called l -models) on arbitrary Cayley tree has been proved in [26, 45, 46].

From the proofs of Theorem 5.1 and 5.2 and analyzing the equation (3.8) we may formulate the following conjecture.

Conjecture 5.3. *Let $p \geq 3$ and $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. Then there exists a unique p -adic Gibbs measure corresponding to the model (3.1) on the Cayley tree of order $k \geq 1$.*

6. THE PHASE TRANSITIONS

In this section, we establish the existence of the phase transition for the p -adic Ising-Vannimenus model (3.1) on a Cayley tree of order k . We need the following auxiliary result.

Proposition 6.1. *Let $\mathbf{u} = (u_1, u_2, u_3)$ be any solution of the system of equations (4.1) and $Z_n^{(\mathbf{u})}$ be a partition function (3.3) associated with \mathbf{u} . Then one has that*

$$Z_n^{(\mathbf{u})} = \left(\frac{a(b^2 u_3 + 1)}{b u_3} \right)^{k|V_{n-1}|}.$$

Proof. We are going to provide some recurrence formula for the partition function $Z_n^{(\mathbf{u})}$. For the given configuration $\sigma \in \Omega_{V_n}$ and $n \geq 1$, we denote

$$(6.1) \quad U_{xy,\sigma}^{(n)} = \begin{cases} 1, & \text{if } \langle x, y \rangle \in \mathcal{N}_{1,n}(\sigma) \\ \frac{a^2}{u_{xy,3}}, & \text{if } \langle x, y \rangle \in \mathcal{N}_{2,n}(\sigma) \\ \frac{a^2}{u_{xy,1}}, & \text{if } \langle x, y \rangle \in \mathcal{N}_{3,n}(\sigma) \\ \frac{u_{xy,2}}{u_{xy,1}}, & \text{if } \langle x, y \rangle \in \mathcal{N}_{4,n}(\sigma) \end{cases}$$

There exists some function $D(x, y) \in \mathbb{Q}_p$ such that

$$(6.2) \quad \prod_{z \in S(y)} \sum_{\omega(z) \in \{\pm 1\}} \exp_p [\omega(z)(J\sigma(y) + J_1\sigma(x))] U_{yz,\sigma \vee \omega}^{(n)} = U_{xy,\sigma}^{(n-1)} D(x, y)$$

where $x \in W_{n-2}$, $y \in S(x)$ and $\sigma \in \Omega_{V_{n-1}}$. It follows from the last equality that

$$\prod_{\substack{x \in W_{n-2} \\ y \in S(x)}} D(x, y) U_{xy,\sigma}^{(n-1)} = \prod_{\substack{x \in W_{n-2} \\ y \in S(x)}} \prod_{z \in S(y)} \sum_{\omega(z) \in \{\pm 1\}} \exp_p [\omega(z)(J\sigma(y) + J_1\sigma(x))] U_{yz,\sigma \vee \omega}^{(n)}$$

Let $U_{n-1} = \prod_{\substack{x \in W_{n-2} \\ y \in S(x)}} D(x, y)$. By multiplying $\exp_p(H_{n-1}(\sigma))$ both sides of last equality, we obtain that

$$\begin{aligned} U_{n-1} \exp_p(H_{n-1}(\sigma)) \prod_{\substack{x \in W_{n-2} \\ y \in S(x)}} U_{xy, \sigma}^{(n-1)} &= \\ &= \exp_p(H_{n-1}(\sigma)) \prod_{\substack{x \in W_{n-2} \\ y \in S(x)}} \prod_{z \in S(y)} \sum_{\omega(z) \in \{\pm 1\}} \exp_p[\omega(z)(J\sigma(y) + J_1\sigma(x))] U_{yz, \sigma \vee \omega}^{(n)} \end{aligned}$$

It follows from (6.1) and (3.9) that

$$(6.3) \quad U_{n-1} Z_{n-1}^{(u)} \mu_u^{(n-1)}(\sigma) = Z_n^{(u)} \sum_{\omega} \mu_u^{(n)}(\sigma \vee \omega).$$

Since $\mu_u^{(n)}$ is a probability measure, i.e. for each $n \in \mathbb{N}$, one has

$$\sum_{\sigma \in \Omega_{V_{n-1}}} \mu_u^{(n-1)}(\sigma) = \sum_{\sigma \in \Omega_{V_{n-1}}} \sum_{\omega \in \Omega_{W_n}} \mu_u^{(n)}(\sigma \vee \omega) = 1,$$

we then find from (6.3) that

$$(6.4) \quad Z_n^{(u)} = U_{n-1} Z_{n-1}^{(u)}.$$

Now assume that $\mathbf{u}_{xy} = (u_1, u_2, u_3)$ for any $\langle x, y \rangle \in L$. It follows from (6.2) with $\sigma(x) = \sigma(y) = 1$ that

$$D(x, y) = \prod_{z \in S(y)} (ab + (ab)^{-1} a^2 u_3^{-1}) = \left(\frac{a(b^2 u_3 + 1)}{b u_3} \right)^k.$$

Consequently, we get from (6.4) that

$$Z_n^{(u)} = \prod_{\langle x, y \rangle \in L_{n-1}} \left(\frac{a(b^2 u_3 + 1)}{b u_3} \right)^k = \left(\frac{a(b^2 u_3 + 1)}{b u_3} \right)^{k|V_{n-1}|}.$$

□

Theorem 6.2. Let $p \geq 3$, $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. Assume that one of the following conditions is satisfied:

(i) $(k, p) = 1$ and $\frac{p-1}{(k, p-1)}$ is even;

(ii) k is odd and $\max\{|k|_p, |a-1|_p\} < |b-1|_p$;

then there exists a phase transition for the p -adic Ising-Vannimenus model (3.1).

Proof. (i) Due to Theorem 4.6, we infer that there is a unique p -adic Gibbs measure μ_0 and d -number of p -adic quasi Gibbs measures μ_r , $r = \overline{1, d}$ where $d = (k, p-1)$. Hence, in order to prove the existence of the phase transition, we have to check for boundedness/unboundedness of these measures.

As we already knew, the boundary functions corresponding to the measures μ_r , $r = \overline{0, d}$ are $\mathbf{u}_{(r)} = (u_1^{(r)}, u_2^{(r)}, u_3^{(r)})$, $r = \overline{0, d}$ in which $\mathbf{u}_{(0)} \in \mathcal{E}_p^3$ and $\mathbf{u}_{(r)} \in (-\mathcal{E}_p)^3$, $r = \overline{1, d}$. Due to Proposition 6.1, we have

$$Z_n^{(\mathbf{u}_{(r)})} = \left(\frac{a(b^2 u_3^{(r)} + 1)}{b u_3^{(r)}} \right)^{k|V_{n-1}|}, \quad r = \overline{0, d}, \quad n \in \mathbb{N}.$$

By using Lemma 2.4, we can easily compute the p -adic norm of $Z_n^{(\mathbf{u}_{(r)})}$ as

$$(6.5) \quad \left| Z_n^{(\mathbf{u}_{(r)})} \right|_p = \begin{cases} 1, & \text{if } r = 0; \\ \leq p^{-k|V_{n-1}|}, & \text{if } r = \overline{1, d}. \end{cases}$$

It follows from (3.9) and $|u_3^{(r)}|_p = 1$, $r = \overline{0, d}$ that

$$\left| \mu_{\mathbf{u}_{(r)}}^{(n)}(\sigma) \right|_p = \frac{1}{\left| Z_n^{(\mathbf{u}_{(r)})} \right|_p}, \quad \forall \sigma \in \Omega_n.$$

Therefore, from (6.5) one finds

$$\left| \mu_{\mathbf{u}_{(r)}}^{(n)}(\sigma) \right|_p = \begin{cases} 1, & \text{if } r = 0; \\ \geq p^{k|V_{n-1}|}, & \text{if } r = \overline{1, d}. \end{cases}$$

This means that the measure μ_0 is bounded and the measures μ_r for all $r = \overline{1, d}$ are not bounded. Therefore, there exists a phase transition for the p -adic Ising-Vannimenus model.

(ii) This case can be proceeded by the same argument as above. This completes the proof. \square

Remark 6.1. Theorems 4.6-4.7 and 6.2 extend all results of [47] to arbitrary order Cayley trees. However, the question for an existence of the strong phase transition for the p -adic Ising-Vannimenus model still remains to be open. We can conditionally prove it by assuming the following conjecture.

Conjecture 6.3. *Let $p \geq 3$ and $a, b \in \mathcal{E}_p$, $b \neq 1$. Let Δ be a set of all solutions of the system (4.1). If $(k, p) = 1$ and $\frac{p-1}{(k, p-1)}$ is even then $\Delta \cap ((\mathbb{Q}_p \setminus \mathbb{Z}_p) \times (-\mathcal{E}_p) \times (-\mathcal{E}_p)) \neq \emptyset$.*

Theorem 6.4. *Let $|J|_p \leq \frac{1}{p}$, $0 < |J_1|_p \leq \frac{1}{p}$. If Conjecture 6.3 holds true then there exists a strong phase transition for the p -adic Ising-Vannimenus model.*

Proof. Let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{Q}_p$ be a solution of the system (4.1) such that $|u_1|_p > 1$. Due to Proposition 4.2, we have that $|u_2|_p = |u_3|_p = 1$ and $|u_3 + 1|_p < 1$. By means of the strong triangle inequality, it follows from the first equation of the system (4.1) that

$$|u_1|_p = \left(\frac{|b^2 u_3 + 1|_p}{|u_3 + b^2|_p} \right)^k > |b^2 u_3 + 1|_p^k.$$

According to Proposition 6.1, from last inequality we obtain that

$$(6.6) \quad \left| Z_n^{(\mathbf{u})} \right|_p = \left(\frac{|a(b^2 u_3 + 1)|_p}{|b u_3|_p} \right)^{k|V_{n-1}|} < \left| u_1^{V_{n-1}} \right|_p.$$

Let $\sigma \in \Omega$ be a configuration such that $\sigma(x) = -1$ for all $x \in V \setminus \{x^0\}$. It then follows from (6.6) and (3.9) that

$$\left| \mu_{\mathbf{u}}^{(n)}(\sigma) \right|_p = \frac{\left| u_1^{-|W_n|} \right|_p}{\left| Z_n^{(\mathbf{u})} \right|_p} < \left| u_1^{-|V_n|} \right|_p.$$

Since $|u_1|_p > 1$ and $|V_n| = \frac{k(k^n - 1)}{k - 1}$, one has that

$$\left| \mu_{\mathbf{u}}^{(n)}(\sigma) \right|_p \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Due to Theorem 6.2, we have for any $r = \overline{1, d}$ that

$$\left| \mu_{u(r)}^{(n)}(\sigma) \right|_p \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

This means that there exists a strong phase transition for the p -adic Ising-Vannimenus model. This completes the proof. \square

7. CONCLUSIONS

In the present paper, we have considered the p -adic Ising-Vannimenus model on the Cayley tree of order k ($k \geq 2$). This model is an analogue of the model studied in [57]. A new measure-theoretical approach is developed, in the p -adic setting, to investigate such a model. We have constructed p -adic quasi Gibbs measures via interacting functions. Such kind of measures exist if the interacting functions satisfy multi-dimensional recurrence equations. The existence of p -adic quasi Gibbs measures are established by analyzing fixed points of a multi-dimensional functional equation. It is shown that the translation-invariant p -adic Gibbs measure is unique. This partially confirms the conjecture formulated in [47]. Finally, we are able to establish the existence of the phase transition for the model depending on the order k of the tree and the prime p . One can observe that the values of the norms (i.e. "absolute values"), in the p -adic setting, are discrete, therefore it is impossible to determine exact values of a critical point for the existence of the phase transition. Moreover, in the field of p -adic numbers there is no reasonable order compatible with the usual order in the rational numbers, and this rises some difficulties in the direction of determination of exact critical points. Note that the methods used in the paper are not valid in the real setting, since all of them based on p -adic analysis and p -adic probability measures.

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